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PASSAGE OF STATIONARY PROCESSES THROUGH LINEAR AND NON-LINEAR DEVICES

A. J. F. Siegert

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PASSAGE OF STATIONARY PROCESSES THROUGH LINEAR AND NON-LINEAR DEVICES[1)]

A. J. F. Siegert

Department of Physics, Northwestern University[2)]

and

The RAND Corporation

Abstract

Many problems in the theory of noise and other random functions can be formulated as the problem of finding the probability distribution of the functional

$$u = \int_0^\infty K(t')\ V\ (x(t'))dt'$$

where $K(t)$ and $V(x)$ are known functions and $x(t)$ is a random function of known statistical properties. The problem of finding the probability distribution of the noise output of a receiver consisting of a filter, a detector, and a second filter is of this type. Methods ~~will be~~ discussed which have led to solutions of this problem in some special cases. In the case of multidimensionally Markoffian $x(t)$ the problem will be shown to be equivalent to an integral equation, which in many cases of interest reduces to a differential equation.

(1) Presented at the Symposium on Statistical Methods in Communication Engineering, Berkeley, August 17-18, 1953.

(2) At present on leave at The Institute for Advanced Study, Princeton, N. J.

1. Prerequisites to any theory of signal detection in the presence of noise are the distribution of observable values in the presence of noise and signal and in the presence of noise alone. Generally, these are not known. Even though the source of noise may be known and the noise may, at the source, have simple and well-known statistical properties [as in the case of thermal and shotnoise], signal and noise have to pass through intricate devices before observation, and are presented to the observer as a random process with statistical properties which are generally much more complicated than those of the original noise. The observed output at any time is a functional of signal and noise, that is a quantity which depends -- in a manner determined by the detection device -- on the values assumed by signal and noise up to the time of observation.

There is at present no systematic theory of the probability distribution of such functionals. It is the purpose of this paper to survey methods which have led to results in some cases of interest in engineering applications. In the case of the quadratic detector, we shall give a form of the result which may lend itself better to actual evaluation. Finally, we will present an approach to the problem which may lead to a more systematic treatment. Our presentation will be limited to the case of noise alone, but usually the generalization to noise and signal does not lead to a much more difficult problem if the case of noise alone has been solved.

2. A linear device is defined as a device which responds to input voltages $v_{11}(t)$, $v_{12}(t)$, . . . with outputs $v_{21}(t)$, $v_{22}(t)$, . . .

in such a way, that the response to a linear combination of inputs $v_1(t) = \sum_k a_k v_{1k}(t)$ with constant coefficients a_k is the output $v_2(t) = \sum_k a_k v_{2k}(t)$. If, furthermore, for any t_o, $v_1(t - t_o)$ becomes $v_2(t - t_o)$, the device responds to a sinusoidal input $e^{i\omega t}$ with an output $Y(\omega)e^{i\omega t}$ and is, therefore, completely described by giving the function $Y(\omega)$ for all circular frequencies ω. Our survey will actually be limited to passive linear devices of this type, but many of the results could be easily generalized. Since an arbitrary input $v_1(t)$ can be written as a Fourier integral the output can be written as

(2.1) $$v_2(t) = \frac{1}{2\pi} \int Y(\omega)\, e^{i\omega t} \int e^{-i\omega t'}\, v_1(t')\, dt'\, d\omega$$ (3)

or, after integration over ω as

(2.2) $$v_2(t) = \int \kappa(t - t')\, v_1(t')dt'$$

where

(2.3) $$\kappa(\tau) = \frac{1}{2\pi}\int Y(\omega)\, e^{i\omega\tau}\, d\omega$$

Sometimes it is useful to write (2.2) with a change of variables as

(2.2') $$v_2(t) = \int \kappa(\tau)\, v_1(t - \tau)d\tau$$

---

(3) Integral signs without limits denote $\int_{-\infty}^{\infty}$

For a passive linear network $Q(\tau)$ must vanish for $\tau < 0$, i.e. all singularities of $Y(\omega)$ must lie in the upper half of the complex plane. We thus have

$$(2.2'') \qquad v_2(t) = \int_0^\infty Q(\tau)\, v_1(t-\tau)\,d\tau$$

A linear network is thus represented by a linear operator, which physically represents the response of the network to a unit impulse at time zero. Defining the function $R(\tau)$ by

$$(2.3) \qquad R(\tau) = \int Q(\tau_1)\, Q(\tau_1 - \tau)\,d\tau_1$$

one can write $|Y(\omega)|^2$ in the form

$$(2.4) \qquad |Y(\omega)|^2 = \int e^{-i\omega\tau} R(\tau)\,d\tau = 2\int_0^\infty \cos\omega\tau\, R(\tau)\,d\tau$$

or one can write $R(\tau)$ in the form

$$(2.5) \qquad R(\tau) = \frac{1}{2\pi}\int e^{i\omega\tau}\, |Y(\omega)|^2\,d\omega$$

3. We will now assume that $v_1(t)$ is statistically known, i.e. that we are given

$$(3.1) \qquad \text{joint prob}\left\{\begin{array}{c} x_1 \leq v_1(t_1) < x_1 + dx_1 \\ x_2 \leq v_1(t_2) < x_2 + dx_2 \\ \text{------------------} \\ x_n < v_1(t_n) < x_n + dx_n \end{array}\right\} = W_n(x_1, t_1; x_2, t_2; \ldots x_n, t_n)\,dx_1\,dx_2 \ldots dx_n$$

for any arbitrary number of arbitrary time instants t_j. We will say that $v(t)$ is stationary, if its statistical description remains unchanged, when the time is shifted, i.e. W_n is unchanged if all t_i's are replaced by $t_j + t_o$. The general problem is to find the corresponding probability description for the output $v_2(t)$. One can give a formal solution of this problem for the general case, but it is of no practical value, since in the only case in which it can be evaluated, the solution can be obtained more simply by other means.

Some statistical properties of v_2 are, however, easily obtained. The average[4] $\bar{v}_2$, e.g., is simply

$$\bar{v}_2 = \bar{v}_1 \int_0^\infty Q(\tau)d\tau = Y(0)\bar{v}_1$$

The covariance of v_2 is given by

$$\overline{v_2(t)\, v_2(t+\theta)} = \iint Q(\tau_1)\, Q(\tau_2)\, \overline{v_1(t-\tau_1)\, v_1(t+\theta-\tau_2)}\, d\tau_1\, d\tau_2 \tag{3.2}$$

thus for processes of zero mean, the normalized correlation function becomes

$$\rho_2(\theta) = \frac{\iint Q(\tau_1)\, Q(\tau_2)\, \rho_1(\theta-\tau_2+\tau_1)\, d\tau_1\, d\tau_2}{\iint Q(\tau_1)\, Q(\tau_2)\, \rho_1(\tau_1-\tau_2)\, d\tau_1\, d\tau_2} = \frac{\int R(\tau)\, \rho_1(\tau-\theta)\, d\tau}{\int R(\tau)\, \rho_1(\tau)\, d\tau} \tag{3.3}$$

where $R(\tau)$ is defined by Eq (2.3) and ρ_1 and ρ_2 are, respectively, the normalized correlation functions of v_1 and v_2. The function $R(\tau)/R(o)$

---

(4) The horizontal bar and the symbol $\langle\ \rangle_{Av}$ will be used interchangeably for the ensemble average.

can thus be interpreted as the correlation function of the output in the limit of infinitely short correlation time of the input. Eq (3.3) can of course also be obtained from the relation between the power spectra $G_1(f)$ and $G_2(f)$ $\left[f = \omega / 2\pi\right]$ of v_1 and v_2, respectively,

$$(3.4) \qquad G_2(f) = \left|Y(2\pi f)\right|^2 G_1(f)$$

by means of the Wiener-Khintchine theorem:[5]

$$(3.5) \qquad \rho_{\substack{1\\2}}(\tau) = \int e^{2\pi i f\tau} G_{\substack{1\\2}}(f)df \Big/ \int G_{\substack{1\\2}}(f)df$$

where G_1 and G_2 are defined for negative argument by $G_{\substack{1\\2}}(-f) = G_{\substack{1\\2}}(f)$. Using Eqs (2.4) and (3.5) we get

$$(3.6) \qquad \rho_2(\tau)\int G_2(f)df = \int e^{2\pi i f\tau}\left|Y(2\pi f)\right|^2 G_1(f)df$$

$$= \int e^{2\pi i f\tau} G_1(f)df \int e^{-2\pi i f\tau'} R(\tau')d\tau'$$

$$= \int R(\tau')\,\rho_1(\tau-\tau')dt' \int G_1(f)df$$

Using this equation, with $\tau = o$, we get

$$(3.7) \qquad \int G_2(f)df = \int R(\tau')\,\rho_1(\tau')d\tau' \cdot \int G_1(f)df\ ,$$

since $\rho_2(o) = 1$ and $\rho_1(\tau)$ is an even function of τ. Eq (3.3) then follows from Eqs (3.6) and (3.7).

(5) c.f. M. C. Wang and G. E. Uhlenbeck, Rev. Mod. Phys. 17, 323, 1945: p 326.

In principle one could evaluate moments of multiple products $\overline{v_2(t_1)\, v_2(t_2)\, \ldots\, v_2(t_n)}$ and obtain at least the characteristic function of W_n for v_2, but the computational labor involved will in general be prohibitive. Approximations can be obtained in the limiting cases of networks whose pass bands are either very wide or very narrow. One then expects that the probability functions of the output can be approximated starting with the probability functions of the input in the first case, and with Gaussian functions in the second case.

The problem of finding W_n for v_2 simplifies greatly when the input is Gaussian. A stationary Gaussian process can be defined in various ways, which are all equivalent. We may e.g. give the characteristic function

$$\left\langle e^{i \sum_k \xi_k v(t_k)} \right\rangle_{Av} = e^{-\frac{\sigma^2}{2} \sum \xi_k \xi_e \rho(t_k - t_e)} \tag{3.8}$$

where $\sigma^2 = \overline{v^2}$ and $\rho(\tau)$ is the normalized correlation function. Alternatively we may state the formula for the moments of products:

$$\left\langle v(t_1)\, v(t_2)\, \ldots\, v(t_{2n}) \right\rangle_{Av} = \sigma^{2n} \sum_{\substack{\text{all} \\ \text{pairs}}} \rho(t_i - t_j)\, \rho(t_k - t_e)\, \ldots\, \rho(t_p - t_q)$$

$$\left\langle v(t_1)\, v(t_2)\, \ldots\, v(t_{2n+1}) \right\rangle_{Av} = 0\,, \tag{3.9}$$

where n is a positive integer.

A linear integral form $\int g(t)\, v(t)\, dt$ of a stationary Gaussian random function $v(t)$ has a Gaussian distribution, as one sees, e.g. by

considering the characteristic function

$$
\begin{aligned}
\left\langle e^{iz\int q(t)\,v(t)dt}\right\rangle_{Av} &= \sum_{0}^{\infty} \frac{(iz)^m}{m!}\left\langle\left(\int q(t)\,v(t)dt\right)^m\right\rangle \\
&= \sum_{0}^{\infty} \frac{(iz)^m}{m!}\int\cdots\int \prod_{1}^{m} q(t_k)\,dt_k \left\langle \prod_{1}^{m} v(t_j)\right\rangle_{Av} \\
&= \sum_{0}^{\infty} \frac{(i\sigma z)^{2n}}{(2n)!}\int\cdots\int \prod_{1}^{2n} q(t_k)\,dt_k \sum_{\substack{\text{all}\\ \text{pairs}}} \prod \rho(t_i - t_j) \qquad (3.10)\\
&= \sum_{0}^{n} \frac{(i\sigma z)^{2n}}{(2n)!}\,\frac{(2n)!}{2^n n!}\left\{\iint q(t_1)\,q(t_2)\,\rho(t_1 - t_2)dt_1\,dt_2\right\}^n
\end{aligned}
$$

since the number of ways of dividing the time instances into different pairs is $(2n)!\,/\,2^n n!$ (There are $(2n)!$ permutations, but 2^n exchanges of the arguments and $n!$ permutations of the factors give no new division into pairs.) We thus get

$$
\left\langle e^{iz\int q(t)\,v(t)dt}\right\rangle_{Av} = e^{-\frac{\sigma^2 z^2}{2}B} \tag{3.11}
$$

with

$$
B = \iint q(t_1)\,q(t_2)\,\rho(t_1 - t_2)dt_1\,dt_2 \tag{3.12}
$$

This shows that the first distribution of $\int q(t-\tau)\,v(\tau)d\tau$ is Gaussian. To show that the output of a linear device is a stationary Gaussian process if the input is, we chose specially for $q(t)$ the function

$$(3.13) \qquad q(t) = z^{-1} \sum_k \alpha_k \, \delta(t_k - t)$$

and obtain

$$(3.14) \qquad \left\langle e^{i \sum_k \alpha_k v_2(t_k)} \right\rangle_{Av} = \left\langle e^{i \sum_k \alpha_k \int q(t_k - \tau) \, v_1(\tau) d\tau} \right\rangle_{Av}$$

$$= \left\langle e^{iz \int q(\tau) \, v_1(\tau)} \right\rangle_{Av} = e^{-\frac{\sigma^2 z^2}{2} B}$$

where

$$(3.15) \qquad B = z^{-2} \iint \sum_{k,\ell} \alpha_k \alpha_\ell \, q(t_k - \tau_1) \, q(t_\ell - \tau_2) \, d\tau_1 \, d\tau_2 \, \rho_1(\tau_1 - \tau_2)$$

Using Eq (3.3) we may write this in the form

$$(3.16) \qquad \sigma^2 B = z^{-2} \sigma_2^2 \sum_{k,\ell} \alpha_k \alpha_\ell \, \rho_2(t_k - t_\ell)$$

with $\sigma_2^2 = \overline{v_2^2}$ and obtain

$$(3.17) \qquad \left\langle e^{i \sum \alpha_k v_2(t_k)} \right\rangle_{Av} = e^{-\frac{\sigma_2^2}{2} \sum \alpha_k \alpha_\ell \rho_2(t_k - t_\ell)}$$

This has the form of (3.[illegible]) and shows, therefore, that the output of the linear network for Gaussian input is again a Gaussian process.

Since we have in this derivation used Eq (3.[illegible]) as definition of the Gaussian process, we can by specializing to q(t) = δ(t) see that Eq (3.[illegible]) follows from Eq (3.[illegible]).

One can use Eqs (3.11) and (3.12) also to obtain the joint distribution of the expansion coefficients of a stationary Gaussian function $v(t)$ in any ortho-normal system of functions $u_\nu(t)$, e.g. in trigonometric functions. Suppose, for instance, that $u_\nu(t)$ are ortho-normal in $(0, T)$ and write

$$c_\nu = \int_0^T v(t)\, u_\nu(t)\, dt$$

We then choose $q(t) = \sum \xi_\nu u_\nu(t)$ and get

$$\left\langle e^{\, i \sum \xi_\nu c_\nu} \right\rangle_{Av} = e^{-\frac{\sigma^2 B}{2}} \tag{3.13}$$

with

$$B = \iint_0^T q(t_1)\, q(t_2)\, \rho(t_1 - t_2)\, dt_1\, dt_2$$

$$= \iint_0^T \sum \xi_\nu u_\nu(t_1) \sum \xi_\mu u_\mu(t_2)\, \rho(t_1 - t_2)\, dt_1\, dt_2$$

$$= \sum \xi_\nu \xi_\mu \iint_0^T u_\nu(t_1)\, \rho(t_1 - t_2)\, u_\mu(t_2)\, dt_1\, dt_2$$

The expansion coefficients are thus Gaussian variables. We get a specially simple result if we choose for the ortho-normal system the solutions of the integral equation

(3.19) $$\lambda_\nu u_\nu(t_1) = \int_0^T \rho(t_1 - t_2)\, u_\nu(t_2)\, dt_2$$

With this choice we have

(3.20) $$\left\langle e^{i\sum \xi_\nu c_\nu} \right\rangle_{Av} = e^{-\frac{\sigma^2}{2}\sum \xi_\nu^2 \lambda_\nu}$$

i.e. the coefficients become independent Gaussian variables with mean zero and mean square $\overline{c_\nu^2} = \sigma^2 \lambda_\nu$. We can thus construct a Gaussian process by writing

$$v(t) = \sigma \sum_\nu a_\nu \sqrt{\lambda_\nu}\, u_\nu(t)$$

where the a_ν are independent Gaussian variables with mean zero and mean square unity.(6) White noise, i.e. a Gaussian noise with infinitely short correlation time, can be represented in any orthogonal system. If $\rho(t)/\int_0^T \rho(\tau)\, d\tau$ approaches $\delta(t)$, any orthogonal set functions $u_\nu(t)$ satisfies the integral equation, with eigenvalues $\lambda_\nu = \int_0^T \rho(\tau)\, d\tau$, so that we can expand white noise in any orthogonal system in the form

(3.21) $$v(t) = D \sum a_\nu u_\nu(t)$$

where the coefficients a_ν are independent Gaussian variables of mean zero and variance unity.

Note that the coefficients would not have become independent if we had expanded $v(t)$ in a Fourier series. For sufficiently large T, however, the correlation between the coefficients becomes very small, so that we can write the Gaussian process over a large time interval as a fourier series with independent Gaussian variables as coefficients.(7)

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(6) K. Karhunen, Ann. Ac Sci. Fennicae, AI, 34, Helsinki, 1946.
M. Kac and A. J. F. Siegert, Ann. Math. Stat. 18, 38, 1947
(7) S. O. Rice, Bell Tel. J. 23, 282, 1944; 25, 46, 1945.

4. If a Gaussian noise passes through a detector, the output of the detector is non-Gaussian. Some properties of this non-Gaussian process can usually be easily obtained.

For instance, if $v_3(t)$ is the detector output defined by(*).

$$v_3(t) = V\left(v_2(t)\right) \tag{4.1}$$

the first probability density $w(v_3)$ of v_3 is obtained as

$$w(v_3) = \int W(v_2)\, \delta\left(v_3 - V(v_2)\right) dv_2 \tag{4.2}$$

where $W(v_2)$ is the probability density of v_2. This can be simplified to

$$w(v_3) = \sum_j W\left(v_2^{(j)}\right) \left|\frac{dV(x)}{dx}\right|_{v_2^{(j)}} \tag{4.3}$$

where the values $v_2^{(j)}$ are the roots of

$$V\left(v_2^{(j)}\right) = v_3 \tag{4.4}$$

if the roots are distinct.

(*) Another way of defining the detector ("envelope detector") is

$$v_3(t) = V\left(\sqrt{v_{2c}^2(t) + v_{2s}^2(t)}\right) \tag{4.1'}$$

where $v_{2c}(t)$ and $v_{2s}(t)$ are obtained by writing $v_2(t)$ as a modulated carrier

$$v_2(t) = v_{2c}(t) \cos \omega_0 t + v_{2s}(t) \sin \omega_0 t$$

and where in practical problems ω_0 can be chosen such as to make $v_{2c}(t)$ and $v_{2s}(t)$ slowly varying functions of t, compared with $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\, \omega_0 t$. Since in the applications the second filter usually will not pass frequencies of order ω_0, the choice between (4.1) and (4.1') is usually irrelevant.

The spectrum of v_3 is obtained by the method of D. O. North by computing first the covariance and the average of v_3:

(4.5) $$\overline{v_3(t_1)\, v_3(t_2)} = V(x_1)\, V(x_1, t_1; x_2 t_2)\, dx_1 dx_2$$

(4.6) $$\bar{v}_3 = W(x)\, V(x)\, dx$$

where W_2 is the second joint probability function of $v_2(t)$. This yields the correlation function from which the power spectrum can be computed by means of the Wiener-Khintchine theorem.

A major problem arises if one asks for the probability functions of the filtered detector output $v_4(t)$

(4.7) $$v_4(t) = \int_0^\infty K(\tau)\, v_3(t - \tau)\, d\tau = \int_0^\infty K(\tau)\, V\left(v_2(t - \tau)\right) d\tau$$

Solutions of this problem have been obtained only in very few special cases. There are of course trivial limiting cases, e.g. if $V(x) = \alpha x$ there is no detector and the two filters can be considered as one linear device, and if $K(t)$ degenerates into a δ- function, there is no second filter. Also, if the bandwidth of the second filter becomes very small $v_4(t)$ will in general become Gaussian again, and the mean and standard deviation of v_4 are usually easily obtained.

A solution for the special case $V(x) = x^2$, v_2 Gaussian was obtained by M. Kac and the author(9), in the sense that the characteristic function of the probability distribution could be expressed by means of the solutions of a certain, not too formidable, integral equation.

---

(9) M. Kac and A. J. F. Siegert, Phys. Rev. 70, 449, 1946 and J. Applied Physics 18, 383, 1947 (The latter paper treats the envelope detector.)
R. C. Emerson, J. Applied Physics, 24, 1168, 1953

The special case $K(t) = 1$ for $0 \leq t \leq 1$, zero otherwise, $V(x) = x^2$ and $x(t) = \int_0^t I(t)\, dt$ where $I(t)$ is white noise (i.e. $x(t)$ is the Wiener function) was treated by R. H. Cameron and W. T. Martin, J. Math. and Physics, 23, 195, 1944.

Actually the preceding considerations already implied a solution of a still more specialized problem of this type, namely the case

(4.8)
$$K(t) = \begin{cases} 1 \text{ for } t < T \\ 0 \text{ for } t > T \end{cases}$$

which is the case of the ideal integrator circuit. Since we had the representation

(4.9)
$$v_2(t) = \sigma \sum_{\nu=0}^{\infty} a_\nu \sqrt{\lambda_\nu}\, u_\nu(t)$$

we have

(4.10)
$$\int_0^T v_2^2(t)\, dt = \sigma^2 \sum_0^{\infty} a_\nu^2 \lambda_\nu$$

with independent normal variables a_ν. Thus the characteristic function becomes

(4.11)
$$\overline{e^{i\sigma^2 \sum a_\nu^2 \lambda_\nu}} = \prod \int e^{i\sigma^2 a_\nu^2 \lambda_\nu - a_\nu^2/2} \frac{da}{\sqrt{2\pi}}$$

$$= \prod_\nu \frac{1}{\sqrt{1 - 2i\sigma^2 \lambda_\nu}}$$

The same result is obtained with arbitrary weighting function $K(t)$, if the numbers λ_ν are defined as the eigenvalues of the integral equation

(4.12)
$$\int K(\tau_1)\, \rho\,(\tau - \tau_1)\, \varphi_\nu(\tau_1)\, d\tau_1 = \lambda_\nu\, \varphi_\nu(\tau)$$

This can be shown by using the concept of "white noise" or -- if one wants to avoid the representation by white noise -- in the following way: The detector input is expanded in the series

(4.13) $$v_2(t - \tau) = \sigma \sum_\nu c_\nu \lambda_\nu \phi_\nu(t)$$

Since the process is stationary the statistical properties of the variables c_ν do not depend on t. The output of the second filter is then

(4.14) $$v_4(t) = \sigma^2 \int K(\tau) \sum_{\mu,\nu} c_\mu c_\nu \lambda_\nu \lambda_\mu \phi_\mu(\tau) \phi_\nu(\tau) d\tau$$

The functions $\phi_\nu(\tau)$ are orthogonal with weight function $K(\tau)$, and can be normalized so that

(4.15) $$\int K(\tau) \phi_\mu(\tau) \phi_\nu(\tau) d\tau = \delta_{\mu\nu}$$

because they are eigenfunctions of the integral equation (4.12). Thus we have

$$v_4(t) = \sigma^2 \sum \lambda_\nu c_\nu^2$$

The statistical properties of the variables c_ν must now be chosen such that $v_2(t)$ is Gaussian with mean zero, unit standard deviation and correlation function $\rho(\tau)$. If we choose the variables c_ν to be independent and Gaussian with $\overline{c}_\nu = 0$, $\overline{c_\nu^2} = 1$ we have

(4.16) $$\overline{v_2(t_1) v_2(t_2)} = \sigma^2 \sum_{\mu,\nu} \overline{c_\nu c_\mu} \phi_\nu(t_1) \phi_\mu(t_2) \lambda_\nu \lambda_\mu$$

$$= \sigma^2 \sum \lambda_\nu^2 \phi_\nu(t_1) \phi_\nu(t_2)$$

The sum on the right-hand side must be shown to be $\rho(t_1 - t_2)$. One sees this by expanding $\rho(t_1 - t_2)$ as a function of t_2 in the orthogonal system $\phi_\nu(t_2)$:

(4.17) $$\rho(t_1 - t_2) = \sum_\nu g_\nu(t_1)\, \phi_\nu(t_2)$$

To determine the coefficients $g_\nu(t_1)$ we multiply both sides by $K(t_2)\phi_\mu(t_2)$ and integrate over t_2.

(4.18) $$\int K(t_2)\, \rho(t_1 - t_2)\, \phi_\mu(t_2) dt_2 = \sum_\nu g_\nu(t_1) \int K(t_2) \phi_\mu(t_2)\, \phi_\nu(t_2) dt_2$$

Using (4.15) on the r.h.s., and (4.12) on the l.h.s. we get

(4.19) $$\lambda_\mu \phi_\mu(t_1) = g_\mu(t_1).$$

We thus have from (4.16)

(4.20) $$v_2(t_1)\, v_2(t_2) = \sigma^2 \rho(t_1 - t_2)$$

An alternative derivation is obtained by writing v_2 as filtered white noise. One then has

(4.21) $$v_2(t - \tau) = \int Q(\theta - \tau)\, v_1(t - \theta) d\theta$$

and

(4.22) $$v_4 = \int K(\tau) d\tau \iint Q(\theta_1 - \tau)\, Q(\theta_2 - \tau)\, v_1(t - \theta_1)\, v_1(t - \theta_2) d\theta_1\, d\theta_2$$

The white noise v_1 is expanded in terms of a set of functions u_ν which are chosen to be the eigenfunctions of the integral equation

(4.23) $$\lambda'_\nu\, u_\nu(\theta_1) = \int \Lambda(\theta_1, \theta_2)\, u_\nu(\theta_2) d\theta_2$$

with

(4.24) $$\Lambda(\theta_1, \theta_2) = \int K(\tau)\, Q(\theta_1 - \tau)\, Q(\theta_2 - \tau) d\tau.$$

Substituting the expansion

$$(4.25) \qquad v_1(t - \theta_1) = D \sum a_\nu u_\nu(\theta_1)$$

above and using the integral equation yields

$$(4.26) \qquad v_4 = D^2 \sum_\nu a_\nu^2 \lambda'_\nu$$

The integral equation can be brought into the previous form by integrating after multiplication with $Q(\tau_1 - \theta_1)$, and substituting

$$(4.27) \qquad f_\nu(\tau_1) = \int Q(\theta_1 - \tau_1)\, u_\nu(\theta_1) d\theta_1$$

We then have

$$(4.28) \qquad \lambda'_\nu f_\nu(\tau_1) = \int Q(\theta_1 - \tau_1) \int K(\tau)\, Q(\theta_1 - \tau)\, Q(\theta_2 - \tau)\, u_\nu(\theta_2) d\tau\, d\theta_2\, d\theta_1$$

$$= \int K(\tau) \left[\int Q(\theta_1 - \tau_1)\, Q(\theta_1 - \tau) d\theta_1 \right] f_\nu(\tau) d\tau$$

The integral in parenthesis is obtained by expressing $\overline{v_2(t_1)\, v_2(t_2)}$ in two ways:

$$(4.29) \qquad \overline{v_2(t_1)\, v_2(t_2)} = \sigma^2 \rho(t_1 - t_2) = D^2 \int Q(t_1 - \theta)\, Q(t_2 - \theta) d\theta$$

[The latter follows from (4.21) and (4.25)]. It follows that $D^2 \lambda'_\nu = \sigma^2 \lambda_\nu$ and we have the former result.

To find the characteristic function of the probability distribution of the output for a network consisting of a quadratic detector thus requires the solution of a homogeneous integral equation and the evaluation of an infinite product. Fortunately both problems are not quite as formidable as they appear.

The infinite product is a Fredholm determinant and can be evaluated in terms of the solution of an inhomogeneous integral equation closely related to the homogeneous integral equation[10]. For completeness we will give here a heuristic derivation in the notation suitable for our purposes. We note first

$$(4.30) \qquad \frac{1}{i}\frac{d}{d\xi}\lg\prod_{\nu}(1-2i\xi\lambda_\nu)^{-1/2} = \sum_{\nu}\frac{\lambda_\nu}{1-2i\xi\lambda_\nu}$$

which can be written as

$$(4.31) \qquad \sum_{\nu}\int\frac{\lambda_\nu}{1-2i\xi\lambda_\nu}\,\phi_\nu(\tau)\,\phi_\nu(\tau)\,K(\tau)\,d\tau$$

we will show that the kernel

$$G(\tau_1, \tau_2; \xi) \equiv \sum_{\nu}\frac{\lambda_\nu}{1-2i\xi\lambda_\nu}\,\phi_\nu(\tau_1)\,\phi_\nu(\tau_2)$$

is essentially the Volterra reciprocal function of our kernel $K(\tau)\,\rho(\tau_1 - \tau)$, if $\phi_\nu(\tau)$ and λ_ν are its eigenvalues and eigenfunctions.
To see this one uses the original integral equation to evaluate the expression

$$(4.33) \qquad G(\tau, \tau_2; \xi) - 2i\xi\int K(\tau_1)\,\rho(\tau_1 - \tau)\,G(\tau_1, \tau_2; \xi)\,d\tau_1$$

and one obtains

$$(4.34) \qquad \sum_{\nu}\frac{\lambda_\nu}{1-2i\xi\lambda_\nu}\left[\phi_\nu(\tau)\,\phi_\nu(\tau_2) - 2i\xi\lambda_\nu\,\phi_\nu(\tau)\,\phi_\nu(\tau_2)\right]$$

$$= \sum_{\nu}\lambda_\nu\,\phi_\nu(\tau)\,\phi_\nu(\tau_2) = \rho(\tau - \tau_2)$$

(10) Whittaker and Watson, [redacted], Cambridge University press, 19[illegible], sec 11.21 examples 1, 2 and sec 11.22.

The function $G(\tau_1, \tau_2, \ )$ is thus the solution of the inhomogeneous integral equation

$$(4.35) \qquad G(\ , \ _2; \ ) - 2i \ K(\ _1) \rho(\ _1 - \ ) G(\ _1, \ _2; \ ) d \ _1 = \rho(\ - \ _2)$$

and the infinite product can be written as

$$(4.36) \qquad \prod (1 - 2i \lambda_{\ }) ^{-1/2} = e^{i \int d\eta \ d\ G(\ , \ ; \eta) K(\ )}$$

since

$$(4.37) \qquad \frac{1}{i} \frac{\ }{\ } lg \prod (1 - 2i \lambda_{\ })^{-1/2} = G(\ , \ ; \ ) K(\ )$$

and, therefore

$$(4.38) \qquad lg \prod (1 - 2i \lambda_{\ }) ^{-1/2} = i \ d\eta \ G(\ , \ , \ ) K(\ ) d$$

Instead of solving the homogeneous integral equation and evaluating the infinite product one can thus solve the inhomogeneous integral equation (4.35) and evaluate the double integral (4.38).

If the first filter is of a lumped circuit network and the input is white noise the integral equation reduces to a differential equation. To show this we note that

$$(4.39) \qquad \rho(t) = 1/2 \ |Y(\omega)|^2 e^{i\omega t} d\omega$$

where $Y(\omega)$ is the response function of the first filter. If the first filter is a lumped circuit network $|Y(\omega)|^2$ can be written as the quotient of two polynomials $P_1(\omega)$ and $P_2(\omega)$

$$(4.40) \qquad |Y(\omega)|^2 = P_1(\omega) / P_2(\omega)$$

where, in order for the problem to be meaningful $P_2(\omega)$ must not be of

lower degree than $P_1(\omega)$. We now form the operators $P_1(-i\frac{d}{dt})$ and $P_2(-i\frac{d}{dt})$ by substituting $(-i\frac{d}{dt})$ for ω in the polynomials. We then get

(4.41)
$$\frac{P_2(-i\frac{d}{dt})}{P_1(-i\frac{d}{dt})}\rho(t) = 1/2 \int Y(\omega)^2 \frac{P_2(\omega)}{P_1(\omega)} e^{i\omega t}\, d\omega$$

$$= 1/2 \int e^{i\omega t}\, d\omega = \pi\,\delta(t)$$

The integral equation thus reduces to

(4.42)
$$\frac{P_2(-i\frac{d}{d\tau})}{P_1(-i\frac{d}{d\tau})} G(\tau, \tau_2, \cdot) - 2i\,\pi K(\tau)\, G(\tau, \tau_2, \cdot) = \pi\delta(\tau - \tau_2)$$

and further to the differential equation

(4.43)
$$P_2(-i\frac{d}{d\tau}) G(\tau, \tau_2; \cdot) - 2i\, P_1(-i\frac{d}{d\tau})\, K(\tau)\, G(\tau, \tau_2; \cdot) = P_1(-i\frac{d}{d\tau})\, \delta(\tau - \tau_2)$$

This means that one has to find two solutions of the linear differential equation

(4.42)
$$(P_2 - 2i P_1 K)\, G = 0$$

valid for $\tau < \tau_2$ and $\tau > \tau_2$, respectively and match these solutions at $\tau = \tau_2$ such that the singularity on the r.h.s. is obtained.

5. The preceding section has been almost exclusively devoted to the evaluation of the characteristic function of the probability distribution of $\int_0^\infty K(t)\, x^2(t)dt$, with $x(t)$ Gaussian, because this is the only case which has been solved for arbitrary $K(t)$ and correlation function.

One case involving a linear detector has been solved by M. Kac [11], who calculated the characteristic function for the probability distribution of $\int_0^1 \left| \int_0^t \xi(\tau)d\tau \right| dt$ where $\xi(\tau)$ is white noise.

Closely related to the problems considered here are the problems of finding the probability distribution of the first passage time, maximum and range of a random function [12]. For instance, the probability that x(t) remains smaller than or equal to a chosen value b in the time interval $(0, \tau)$ is the probability that $\int_0^t V(x(t))\,dt = 0$, with $V(x) = 0$ for $x \le b$ and $V(x) = 1$ for $x > b$, since with this choice of V(x) the integral $\int_0^\tau V(x(t))\,dt$ is the total time during which $x(t) > b$.

Problems of this type have been solved in special cases. The common feature of these special cases which made the solution possible is that these are cases with Markoffian x(t). In the engineering problems under consideration here x(t) will not in general be Markoffian but will often be a component of a vector function which is Markoffian [13].

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(11) M. Kac, Trans. Am. Math. Soc. 59, 401, 1946

(12) P. Erdös and M. Kac, Bull. Am. Math. Soc. 52, 292, 1946
A. J. F. Siegert, Phys. Rev. 81, 617, 1951 and papers quoted there.
W. Feller, Ann. Math. Stat. 22, 427, 1951
D. A. Darling and A. J. F. Siegert, RAND P-238 Revised and Ann. Math. Stat. (in press)

(13) M. C. Wang and G. E. Uhlenbeck, ref 5 section 3. An example (output of R, L, C circuit with white noise input) is treated in detail in section 10 of ref.

We will now show how the general problem of finding the characteristic function of $\int_0^\infty k(t)\, V\left(x(t)\right) dt$ reduces to the problem of solving an integral equation if $x(t)$ is the projection of a multidimensional Markoffian process, and how, in certain cases of interest, the integral equation can be further reduced to a partial differential equation[14]. We assume that $x(t)$ is a component of a stationary[15], continuous m-dimensional Markoffian process, i.e. that $x(t)$ is either Markoffian itself or that there are m-one functions $v^{(1)}(t), v^{(2)}(t), \ldots, v^{(m-1)}(t)$ such that the vector process $x(t), v^{(1)}(t), \ldots, v^{(m-1)}(t)$ is Markoffian. We will write the following equations for the two dimensional process only but they can be immediately generalized.

We use the notation

(5.1) $$\text{joint prob}\begin{Bmatrix} x \leq x(t) \leq x + dx \\ v \leq v(t) \leq v + dv \end{Bmatrix} = W(x, v)\,dx\,dv$$

and

(5.2) $$\text{conditional prob}\begin{Bmatrix} x \leq x(t) \leq x + dx \\ v \leq v(t) \leq v + dv \end{Bmatrix} \text{ if } \begin{matrix} x(t_0) = x_0 \\ v(t_0) = v_0 \end{matrix} = P(x_0, v_0 \mid x, v, t - t_0)\,dx\,dv$$

In order to obtain the desired characteristic function

(5.3) $$\phi(z) = \left\langle \exp\left(-z \int_0^\infty K(\tau)\, V\, x(\tau)\, d\tau\right)\right\rangle_{Av}$$

[14] A. J. F. Siegert, Two Integral Equations for the Characteristic Functions of Certain Functionals of Multidimensional Markoffian Processes (Mimeographed).
D. A. Darling and A. J. F. Siegert, RAND Report P-429.

[15] The assumption of stationarity could be easily eliminated.

we will first consider the function

(5.4)

$$R(x_0, v_0 | x, v; t) = \left\langle \exp - z \int_0^t K(\tau)\, V(x(\tau))\, d\tau \cdot \sigma(x - x(t))\, \sigma(v - v(t)) \right\rangle_{Av}^{x_0, v_0}$$

where $\langle \quad \rangle_{Av}^{x_0, v_0}$ denotes the conditional average taken with the restriction $x(o) = x_0$, $v(o) = v_0$. We now obtain an integral equation for R by interpreting $\{x(\tau), v(\tau)\}$ as the path of a particle which at time $\tau = 0$ is located at x_0, v_0 and is at that time white, and which moves in a medium which can make it black. The expression $z\, K(t)\, V(x)\,dt$ is for the present assumed non-negative[16] and is interpreted as the probability that the particle becomes black in the time interval $(t, t + dt)$, if it is located at $\{x, v\}$[17]. The probability of blackening in the time interval (o, t) if the particle moves on a fixed path $\{x(\tau), v(\tau)\}$ is then given by $e^{-z \int_0^t K(\tau)\, V(x(\tau))\, d\tau}$. The expression $R(x_0, v_0 | x, v, t)\,dx\,dv$ is, therefore, the probability that the particle is found at time t in the rectangle $\{x, x + dx; v, v + dv\}$ and is still white, if it started at $\tau = 0$ white at x_0, v_0. An integral equation for R is obtained by writing the probability $P(x_0, v_0 | x, v; t)\,dx\,dv$ that the particle is at t in the rectangle $\{x, x + dx; v, v + dv\}$ as the sum of the probability that the particle is still white, and probability that it is black. The latter probability is the probability that it was blackened at some intermediate time t' while in the rectangle $\{x', x' + dx'; v', v' + dv'\}$ and then

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(16) This assumption can be eliminated by a more formal derivation of the results.

(17) We could have worked with the more general expression $\phi(x, v, t)$ instead of the special form $k(t)\, V(x)$.

moved black to be at t in $\{x, x + dx; v, v + dv\}$. We thus have

(5.6) $$P(x_o, v_o \mid x, v; t) = R(x_o, v_o \mid x, v, t)$$

$$+ z \int_0^t K(t')dt' \int R(x_o\, v_o \mid x', v'; t')\, V(x')\, P(x', v' \mid x, v, t - t')dx'\, dv'$$

This is to be considered as an integral equation for $R(x_o, v_o \mid x, v; t)$ in the variables x, v, t, where x_o and v_o enter only as fixed parameters. Since we are interested only in

(5.7) $$f(z) = \lim_{t \to \infty} \int W(x_o, v_o)\, R(x_o\, v_o \mid x, v; t)dx_o\, dv_o dx\, dv$$

we can eliminate the parameters x_o and v_o by multiplying both sides of equation by $W(x_o, v_o)dx_o\, dv_o$ and integrating over x_o, v_o, obtaining thus

(5.8) $$W(x, v) = R(x, v; t)$$

$$+ z \int_0^t K(t')dt' \int R(x', v'; t')\, V(x')\, P(x', v' \mid x, v, t - t')dx'\, dv'$$

with $R(x, v; t) = \int W(x_o, v_o)\, R(x_o\, v_o \mid x, v; t)dx_o\, dv_o$

If P satisfies a Fokker Planck equation

(5.9) $$LP = \frac{\partial P}{\partial t}$$

with initial condition

(5.10) $$P(x_o, v_o \mid x, v, 0) = \delta(x - x_o)\, \delta(v - v_o) ,$$

where L is a differential operator acting on x and v, the integral equation reduces to a partial differential equation, since we get by operating with $L - \frac{\partial}{\partial t}$ on Eq (5.6):

$$(5.11) \qquad 0 = \left(L - \frac{\partial}{\partial t}\right) R(x_0, v_0 \mid x, v, t)$$

$$- zK(t) \int R(x_0, v_0 \mid x', v') V(x') R(x', v' \mid x, v, o) dx' dv'$$

or

$$(5.12) \qquad \left(L - zK(t) V(x) - \frac{\partial}{\partial t}\right) R(x_0, v_0 \mid x, v, t) = 0$$

For $R(x, v; t)$ one obtains the same differential equation. The initial conditions are obtained from the integral equations as

$$(5.13) \qquad R(x_0, v_0 \mid x, v; 0) = \delta(x - x_0) \, \delta(v - v_0)$$

and

$$(5.14) \qquad R(x, v, 0) = W(x, v) ,$$

respectively.

If $x(t)$ is itself Markoffian, the variables v_0 and v do not occur and the operator L operates on x only, otherwise the derivation is unchanged. For the special case of the Wiener process in one dimension, and with $K(t) = 1$ for $0 < t < 1$, zero otherwise a different derivation of equations corresponding to (5.6) and 5.12) was given by M. Kac [18]. We do not know as yet whether the above considerations will help to increase appreciably the number of cases for which an exact solution of our problem is feasible. Even if this is not the case we believe that it is desirable to have one method to derive the few existing solutions which at the present seem to require a special method for each case.

For practical purposes it may be of interest to consider the integral

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[18] M. Kac, Proc. of the Second Berkeley Symposium on Math. Stat. and Prob., 1951, U. of California Press, p 189-215.
Closely related is the Kramers equation, H. A. Kramers, Physica 7, 284, 1940.

equation

$$(5.15) \qquad R_0(x_0, v_0 \mid x, v, t) = R(x_0, v_0 \mid x, v, t)$$

$$+ z \int_0^t dt' \, R(x_0, v_0 \mid x', v', t') \left[K(t') \, V(x') - K_0(t') \, V_0(x') \right] \cdot$$

$$\cdot \, R_0(x', v' \mid x, v, t - t') \, dx' \, dv'$$

which relates the solution $R(x_0, v_0 \mid x, v, t)$ of (5.6) to the solution $R_0(x_0, v_0 \mid x, v, t)$ of the integral equation obtained from (5.6) by replacing $K(t') \, V(x')$ by $K_0(t') \, V_0(x')$. This integral equation can be obtained by an argument quite similar to the one used to obtain (5.6) but involving a mixture of two blackening media, with blackening probability $zK_0(t) \, V_0(x) \, dt$ and $z \left[K(t) \, V(x) - K_0(t) \, V_0(x) \right] dt$, respectively. R and R_0 are interpreted respectively as the probabilities of finding the particle at t in $(x, x + dx; v, v + dv)$ in the presence of both media and in the presence of the first medium alone. If $R_0(x_0, v_0 \mid x, v, t)$ is known, (5.15) can be used -- by successive approximations -- to obtain approximations for $R(x_0, t_0 \mid x, v, t)$ for cases in which $\left| K(t) \, V(x) - K_0(t) \, V_0(x) \right|$ is sufficiently small. We thus have a perturbation method which enables us to obtain e.g. the effect of deviations from the quadratic detector law on the probability distribution of the noise output by successive approximations.